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Using the Logistic Function to Illustrate Periodic Orbits as Recurrent Formation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

Recurrence as behaviour in the dynamical system is also a state which happens as a result of an outcome of a point which does return to its initial state. But in dynamical system the most simple and easy way to study regime in one dimension is the logistic function. In this study we seek to understudy the recurrence as a full strong state from the nature or behaviour of the logistic function, where the periodic orbits as behaviour the logistic function is considered. A point can only be termed as recurrent if it is in its own future state. A periodic orbit returns infinitely often to each point on the orbit. And so it is clear that an orbit is recurrent when it returns repeatedly to each neighbourhood of its initial position. Recurrence as in dynamical system is a result of periodic formation which is a movement that returns back to the original state or position at a constant rate. A systematic example for each periodic point from the logistic function relative to a control parameter λ is discussed. Different iterations tables, diagrams (graphs) for x_n against n, tables of stability of periodic nature which shows relative range of the control parameter λ and x_0 are discussed. Through graphical illustrations and algebraic approach, the study showed that in the formation of recurrence through logistic function, the parameter λ played a major role and not all the periodic points

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(orbits) lead to recurrent formation. The study also showed that unstable behaviour of the logistic function when$\lambda = 4$, ends the periodic behaviour, hence the absence of recurrence. And the absence of recurrent and the unstable nature of system bring about chaos.

Keywords: Periodic; recurrent; logistic; formation; stable; chaos; discrete.

1 Introduction

This particular section talks about a brief explanation of dynamical system, recurrence, iteration, logistic function, definitions and examples.

1.1 Dynamical system

Is simply a system that depends on a rule for its evolution basing its changes on the initial condition for a specific time. This rule which is based on its states is that, it takes or maps a space (phase space) to itself for or within a time. The main objective of dynamical system theory is to study the long-term behaviour of the orbits of the system. Also according to [1], the phase space gives a better platform for the build-up of dynamical system, where it is modelled through time base on a given rule for its evolution. And this system is made up of continuous system which is basically about differential equations and the discrete system which study the maps or difference equations.

Arild Wikan in [2] makes a valid point concerning the concepts of dynamical system, that its discrete part which leads to the study of difference equations and maps has not been given much attention as compare to the continuous part.

Definition 1.0: Dynamical system is (X,n, T), where X is a set (phase space), n is either $\mathbb{Z}$ or $\mathbb{R}$, and T is an operator of X (with discrete time if n=$\mathbb{Z}$) as defined in [3].

That is, $T: X \rightarrow X$, where T as an operator depends on time. X is a state space that maps to itself.

Hence, $[T, X]$, is what we called the dynamical system.

Note: The discrete nature of the dynamical system is defined by, $T^n(x) = x$, n=1,2,…, $T^1(x) = x, T^2(x) = T\big(T(x)\big) = T^1(x) = x,\ T^3(x) = T\big(T^2(x)\big) = T^1(x) = x, \ldots\ T^n(x) = x$ the nth system.

Therefore as in [1], if $x \in X$ is an initial state, then the sequence , $x, T^1(x), T^2(x)$, ...,forms the orbits and $T^n(x) = ToTo, \ldots T(x)$ is the nth iterate of the system.

So dynamical system in it totality is the study of how things change over a period of time.

Example 1.0: Populations growth, changes in the weather system etc.

1.2 Recurrence

The George Birkhoff Recurrence theorem motivated several definitions. By His idea, an important role in the proof is played by sets which get mapped into themselves under the transformation.

In [4], Simon makes a valid point that, in dynamical system, the discrete dynamical system can also be defined in a recursion form as; $x_{n+1} = f(x_n)\ \ n\ =\ 0, 1, 2, \ldots$

So as we base our argument on the discrete nature and as we continuously iterate a system, it returns to its original state, then what is formed is called the recurrence.

"The idea of recurrence is that every state, as it evolves forward in time, in some sense eventually return to its original state" as explained vividly in [5].

Definition 1.1: If (X, T) is a dynamical system, a point $x \in X$ is said to be recurrent if for any neighbourhood U of$x$, there exists an integer $n \geq 1$such that $T^n(x) \in U$

Example 1.1: if$h: \mathbb{R} \rightarrow \mathbb{R}$, defined by $h(x) = -x$ implies, $h^n(x) = (-1)^n x$, where n= 1, 2… Hence { $-x$, x, …, $(-1)^n x$ } is the trajectories that keeps on oscillatory between $-x$ and x therefore making them a recurrent points in their isolations.

1.3 Logistic function

The logistic function which is a quadratic map do not stay fixed in its transformation as the parameter λ which control it keep changing. This function as a quadratic function is a difference equation which is a non-linear system, given as,

$$x_{n+1} = \lambda x_n(1 - x_n)$$

$x_{n+1} = \lambda x_n - \lambda {x_n}^2$, where $n = 0,1,2, \ldots$, which gives the discrete time and $\lambda \in [1,4]$ is the control parameter and $x \in [0,1]$.

1.4 Iteration

Note: If we let, $x_{n+1} = f(x)$, then $f(x) = \lambda(x - x^2) = \lambda x(1 - x)$

If $f: x \rightarrow x$ or$x_n \rightarrow x_{n+1}$, where x_{n+1} is the sequence of points generated.

If $n = 0$, $x_1 = f^1(x_0)$, then $x_n = f^n(x_0)$

Hence, $x_0, x_1, x_2, x_3, \ldots.$ is the sequence which is called the orbits of $x_{n+1} = f(x)$.

Remarks: $x_0, x_1 = f(x_0), x = f^2(x_1)$,…… are the orbits or trajectory. Iteration is the process where a seed (initial point) is put into a rule then the outcome is used as a new input continually.

Example 1.3: Given f(x) = $\sqrt{x}$, then

X_0=6561

X_1= $f(x)$ =$\sqrt{6561} = 81$

X_2=$f^2(x_1)$ =$\sqrt{81} = 9$

X_3=$f^3(x_2)$ =$\sqrt{9} = 3$

X_4=$f^4(x_3)$ =$\sqrt{3} = 1.73$

x_0, $f(x_0)$, $f^2(x_1)$, ………….,$f^n(x_{n-1})$

6561, 81, 9, 3,………….$f^n(x_{n-1})$

Note: Clearly, as iteration occurs based on evaluating a particular function over with its outputs, the outputs obtained from each evaluation form what we call the Orbit in sequential order.

Definition 1.3: let$f$: $X \rightarrow X$ the point x_0 is a ***periodic point*** of period n for f if $f^n(x_0) = x_0$ where the point x_0 is ***fixed point*** for f if $f^i(x_0) \neq x_0$ for all i. The sequence $\{x_0, f(x_0), f^2(x_0) \ldots f^n(x_0) \ldots\}$ is called the orbit of x_0 under f.

If $f: x \rightarrow x$ or $x_n \rightarrow x_{n+1}$ and $f(x) = \lambda(x - x^2)$

Then as explained in [6], x_0 as a fixed point, is;

1. Attracting (sink) if $|f'(x_0)| < 1$
2. Repelling (source) if $|f'(x_0)| > 1$

And from [7], taking the derivative of the function$f$, the stability of periodic point is exist if $|f'(x_0)| < 1$ is true, then x_0 as a fixed point is stable and instability of the periodic point is when$|f'(x_0)| > 1$, then x_0 as a fixed point is unstable. Hence period n gives rise to stability of fixed point f^n when there is an entire attraction of the neighbourhood.

According to [8], the fixed point $x_0 = 0$, $\lambda \epsilon [0,1], ie - 1 < \lambda < 1$ which is within the domain of $\lambda \epsilon [1,4]$ is attracting (which then make it stable) as a period- 1 orbit but at $\lambda \epsilon (1,4)$, the function is repelling making it unstable. At the fixed point $x_0 = \frac{\lambda - 1}{\lambda}$, the function is asymptotically stable at $1 \leq \lambda \leq 3$ hence attracting fixed point and repelling at $x_0 = \frac{\lambda - 1}{\lambda}$ when$\lambda \epsilon (0,1) or\ (3,4)$.

2 Main Work

Recurrent formation from the first periodic point (fixed point), with different values for the control parameterλ. Here we try to find out from the two behaviour of the fixed point that is attracting and repelling which one is recurrent in nature. Upon the whole, with the help of graphical illustrations [9], this will denote better demonstrations and clarity.

Now at $\lambda = 1.1$ and placing this 1.1 in $x_0 = \frac{\lambda - 1}{\lambda} = \frac{1}{11}$, the iteration of the function $f(x) = \lambda(x - x^2)$ gives the same value 0.09090909091 which is a repetition just after the first decimal value which is also the same as the initial value.

Table 1. Iteration of $f(x) = \lambda(x - x^2)$ with $\lambda = 1.1$ and $x_0 = 0.090900$

n	**0**	**1**	**2**	**3**	**4**
x_n	0.090900	0.090901	0.090902	0.090903	0.090904

Hence several iterations will yield the same value. Therefore, when a control parameterλ, is placed into $x_0 = \frac{\lambda - 1}{\lambda}$ the outcome of it by manipulation forms orbits which are the same as the original value x_0 making it continues, equilibrium (Fig. 1) and stable (Table 2) at this value $x_0 = \frac{1}{11}$ hence recurrence since constant iteration comes back to it original state.

Table 2. Stability of the periodic nature when $\lambda = 1.1\ and$ and $x_0 = 0.090900$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.0909091	Stable

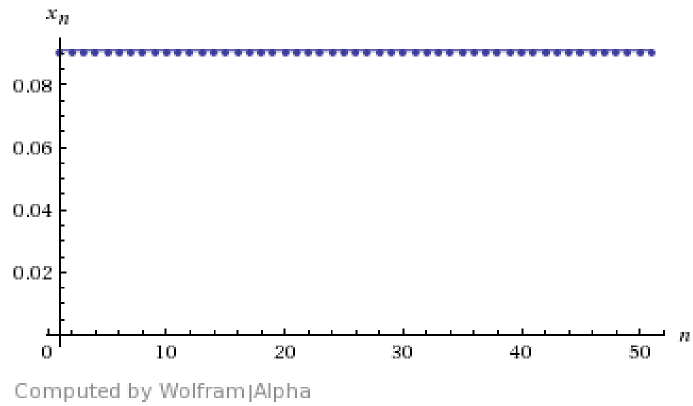

Fig. 1. The equilibrium nature of $f(x) = \lambda(x - x^2)$ with $\lambda = 1.1$ and $x_0 = 0.090900$

2.1 Illustration of period-1 as recurrent formation

Example 1: Illustration of the attracting fixed point of the logistic function as a period- 1 recurrent formation when $\lambda < 3$ and then taking a fixed point which is not dependent on λ. That is if $\lambda = 2.5 < 3$ and $x_0 = 0.20$

$$f(x) = \lambda(x - x^2) = 2.5(x_n - {x_n}^2) \text{ , at} x_0 = 0.20,$$

Table 3. Iteration of$f(x) = \lambda(x - x^2) = 2.5(x_n - {x_n}^2)$, at$x_0 = 0.20$,

n	0	1	2	3	4
x_n	0.20000	0.40000	0.60000	0.60000	0.60000

It is very obvious that taking a control parameter $\lambda < 3$ and $x\epsilon[0,1]$ the successive iteration gives orbits or sequence which is converging and has the ability to return to one specific value. Now per what we have shown in Table 3 above and Fig. 2, the first two outcomes form the transient and when we dump/cut them off, the rest of the orbits or the trajectory forms recurrence hence (period-1 recurrent) that is {…,0.60, 0.60, 0.60, …}

Table 4. The stability of periodic nature when $\lambda = 2.5$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.60	Stable

Clearly, as the function returns to a specific value after several iterations, such outcome is continues, equilibrium and stable as shown in (Fig. 2), (Table 3) and (Table 4). Hence this point of convergence is what form the period-1 recurrent.

Example 2: Illustration of the repelling fixed point of the logistic function as whether or it gives period-1 recurrent, when $\lambda > 3$ and $x \,\epsilon[0,1]$.

Let $\lambda = 3.1 > 3$ then $x_0 = \frac{3.1-1}{3.1} = \frac{21}{31} = 0.677$ as the initial point and also as the fixed point.

$$f(x) = \lambda(x - x^2) = 3.1(x_n - {x_n}^2)$$

First we take a different initial value and then iterate the function and see what happens.

$x_0 = 0.10000$

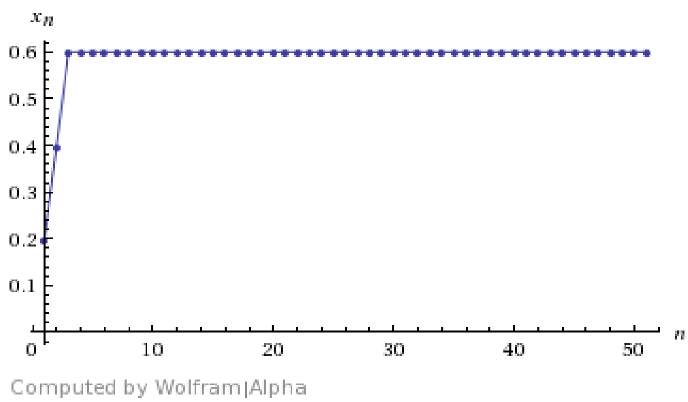

Fig. 2. The period-1 uniformity of $f(x) = \lambda(x - x^2)$ with $\lambda = 2.5$ and $x_0 = 0.2000$

Table 5. Iteration of$f(x) = 3.1(x_n - {x_n}^2)$, at$x_0 = 0.10000$

n	0	1	2	3	4
x_n	0.10000	0.27900	0.62359	0.72765	0.61435

As we keep on iterating the function, it also keep moving away from the fixed point 0.677, these repelling trajectories {0.10000, 0.27900, 0.62359, 0.72765, 0.61435, …}on both left and right of the fixed point gives different outcomes and never shows a formation of period-1 recurrent.

But when we iterate the function using the fixed point 0.677 as the initial value, it gives out slightly unchanged values which when consider approximation forms period-1 recurrent because it tends to be diverging gradually from the fixed point on both its left and right as we keep iterating as shown in Table 6 and Fig. 3.

That is, $x_0 = 0.677$

Table 6. Iteration of $(x) = 3.1(x_n - {x_n}^2)$, at$x_0 = 0.67700$,

n	0	1	2	3	4
x_n	0.67700	0.67788	0.67691	0.67798	0.67681

Orb {0.6770, 0.6778, 0.6769, 0.6780, 0.6768, 0.6781, 0.6767, 0.6767…}.

Hence unstable at parameter $\lambda > 3$ for period-1 as shown in the Fig. 3 and Table 7.

Table 7. The stability of the periodic nature when $\lambda = 3.1$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.677419	Unstable
2	0.558014,0.764567	Stable

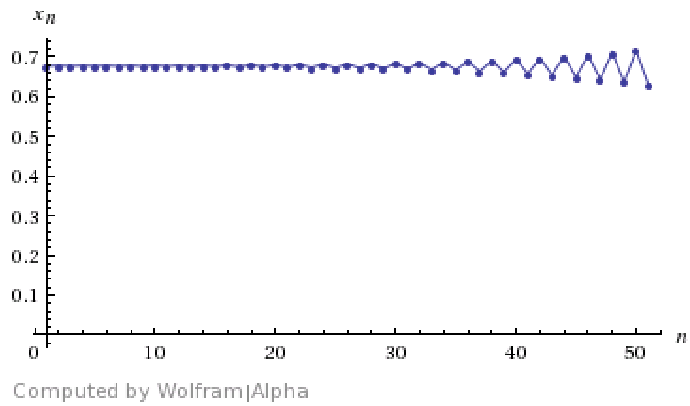

Fig. 3. Graph of $(x) = \mathbf{3.1}(x_n - x_n{}^2)$, at$x_0 = \mathbf{0.67700}$

Now taking a parameter higher than 3.1 like λ=3.5 > 3 with a fixed iterative point $x_0 = 0.71$, the function diverges totally from the repelling fixed making it unstable with no traces of recurrent as shown in Table 7, Fig. 4 and Table 8.

Also at $x_0 = 0.71$

Table 8. Iteration of$f(x) = \lambda(x - x^2) = \mathbf{3.5}(x_n - x^2{}_n)$, at $x_0 = \mathbf{0.7100}$,

n	0	1	2	3	4
x_n	0.7100	0.72065	0.70460	0.72849	0.69227

Table 9. Stability of the periodic nature at $\lambda = \mathbf{3.5}$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.0714286	Unstable
2	0.428571,0.857143	Unstable
4	0.38282,0.856941,0.500884,0.874997	Stable

One can now say that period-1 recurrent do occur/exist at the attracting fixed point of the logistic function within the ranges $-1 < \lambda < 1$ and $\lambda < 3$ but at the repelling fixed point of the logistic function period-1 recurrent tends to occur depending on both the control parameter $\lambda\epsilon(0,1)or\ (3,4)$ and the initial point hence when a fixed point is repelling, period-1 recurrent formation is not totally assured.

2.2 Illustration of period-n as recurrent formation

Here we will consider the period-2 and period-3 to see the existence of all periodic recurrent formation.

If $f: x \to x$ or $x_n \to x_{n+1}$

Then,$f(x) = \lambda(x - x^2)$, produces period-2 orbits if after several iteration the outcome of the function alternate between two values. Under this condition, the control parameter plays a very vital role as it takes the range$\lambda \geq 3$. Recurrent as we have seen above is the ability of system to return to its original state after passing through a specific time. So when the function $f(x)$ is iterated with a seed which emit second value and upon successive iteration the outcome only the seed and the emitted second value then period-2 recurrent is formed.

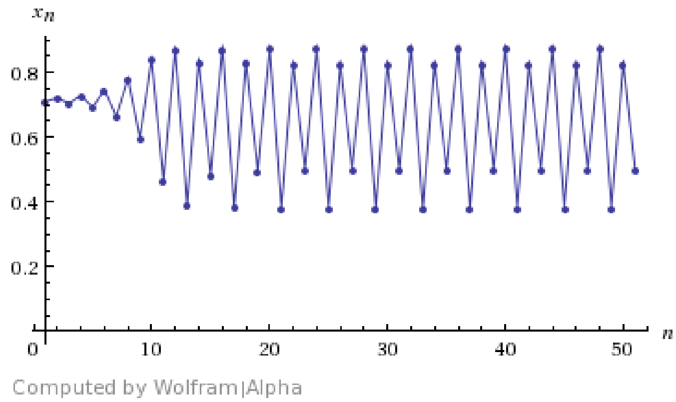

Fig. 4. Graph of the repelling point of $f(x) = \lambda(x - x^2) = 3.5(x_n - {x_n}^2)$, at $x_0 = 0.7100$

Example 3: Illustration of the period-2 orbit of the logistic function as a period-2 recurrent.

If $\lambda = 3.2$ and using an initial condition of 0.5 on the function $f(x) = \lambda(x - x^2) = 3.2(x_n - {x_n}^2)$

At $x_0 = 0.50000$

Table 10. Iteration of $(x) = \lambda(x - x^2) = 3.2(x_n - {x_n}^2)$, at $x_0 = 0.50000$,

n	x_n
0	0.50000
1	0.80000
2	0.51200
3	0.79954
4	0.51288
5	0.79947
6	0.51302
7	0.77946
8	0.51304
9	0.79946

Graphically when the function was iterated 50 times successively using an initial condition of 0.50, it begins to fluctuate between two values 0.51 and 0.80. These two trajectories forms the period-2 recurrent since after the initial condition the next outcomes (trajectories) moves back and forth on the same two values making the system stable as shown in the Table 11.

Table 11. Stability of the periodic nature at $\lambda = 3.2$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.6875	Unstable
2	0.513045,0.799455	Stable

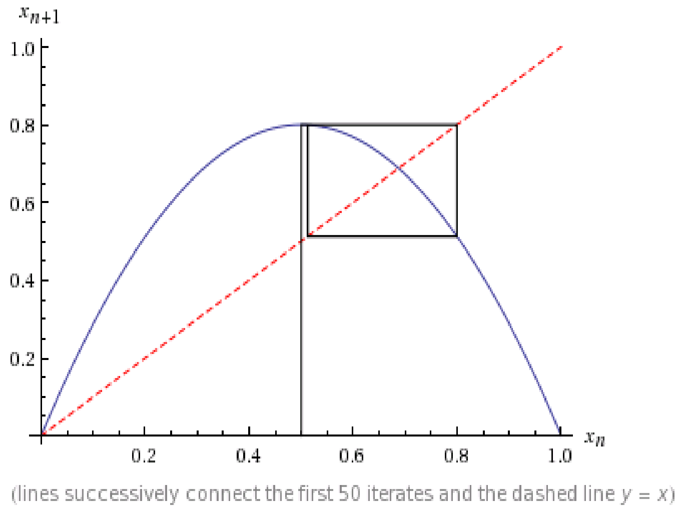

Fig. 5. Graphical display of the iteration of $f(x_n) = 3.2(x_n - {x_n}^2)$

Therefore period-2 recurrent do occur as the results of the function keep returning to its original values as a period-2 cycle as shown in the Fig. 6.

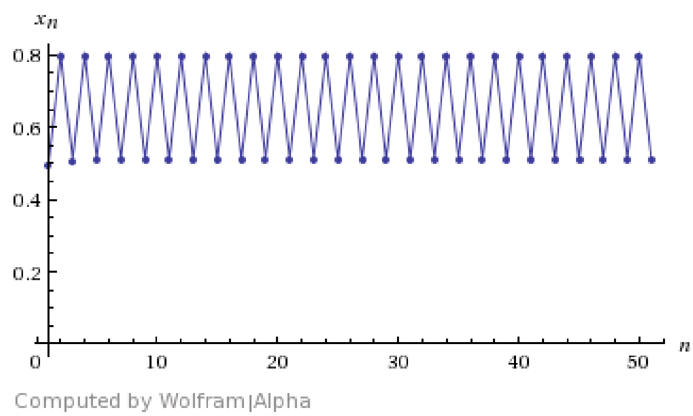

Fig. 6. The period- 2 cycle of the logistic function at $x_0 = 0.50000$, when $\lambda = 3.2$

Again according to [8], as the parameter λ is been increases beyond 3 different periodic orbits are formed. Especially beyond 3.45, new periods which show double behaviours are seen and these double behaviour means they return to their original states. Therefore as the control parameter λ is altered beyond 3 and 3.45 periodic doubling recurrent are forms.

Also [10], showed that period-3 starts when $\lambda \approx 3.83$ when clearly means that at this point three distinct values are expected to keep surfacing irrespective of the number of iterations.

Example 4: Illustration of period-3 orbit of the logistic function as period-3 recurrent.

If $\lambda = 3.83$ and using an initial condition of 0.5 on the function $(x) = \lambda(x - x^2) = 3.83(x_n - {x_n}^2)$, at x_0 0.5.

Table 12. Iteration of $(x) = 3.83(x_n - {x_n}^2)$, at$x_0 = 0.50000$,

n	x_n
0	0.50000
1	0.95750
2	0.15586
3	0.50390
4	0.95744
5	0.15606
6	0.50443
7	0.95742
8	0.15612
9	0.50459

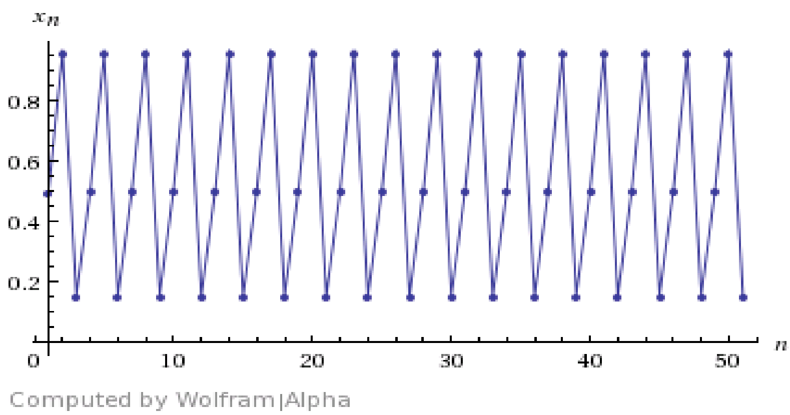

Fig. 7. The period-3 cycle of the logistic function when$\lambda = 3.83$, $x_0 = 0.50000$

The sequence or the trajectories forming out of this is a repeat of values that moves through three numbers {0.50, 0.96, 0.16 …} as shown in the Fig. 7.

Therefore during this period, the system is in a stable state hence period-3 recurrent exist at $\lambda = 3.83$ as it has been shown in Table 13.

Table 13. Stability of the periodic nature when $\lambda = 3.83$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.738903	Unstable
2	0.369161, 0.891935	Unstable
3	0.156149, 0.504666, 0.957417	Stable
3	0.16357, 0.524001, 0.955294	Unstable
4	0.299162,0.803014,0.60584,0.914596	Unstable

Recurrent formation as in dynamical system is indeed dependent on the rule and the parameter given a range is as important as the general rule. Within a specific range of λ the system becomes stable and unstable as the periodic cycle is formed when one keep changing the parameter.

Periodic cycle existence prevent the formation of chaos, hence recurrent absence brings about chaos.

2.3 Illustration of chaos as a result of unstableness of the system at $\lambda = 4$ with $x_0 = 0.50000$

As it was easy for us to describe the period-1 cycle as the period-1 recurrent and the period-2 and period-3 cycle as the period-n recurrent given a parameter, one can not describe the Table 14, Figs. 8, 9 and Table 15 since the system is totally unstable as its approaches infinity. We can see that the uniformity function as the tables and the graphs shows above is totally zero. Fig. 9 and Table 15 shows how difficult it is to read it movement making it uncorrelated and unstable respectively, hence un-periodic which make it to be in the chaotic range. Therefore if the system behaves un-periodic then there is no formation of recurrent. Hence recurrent in it natural form can be term as periodic irrespective of the type of periodic cycle.

Table 14. Iteration of $(x) = 4(x_n - x_n{}^2)$, at $x_0 = 0.50000$,

n	x_n
0	0.50000
1	0.10000
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0

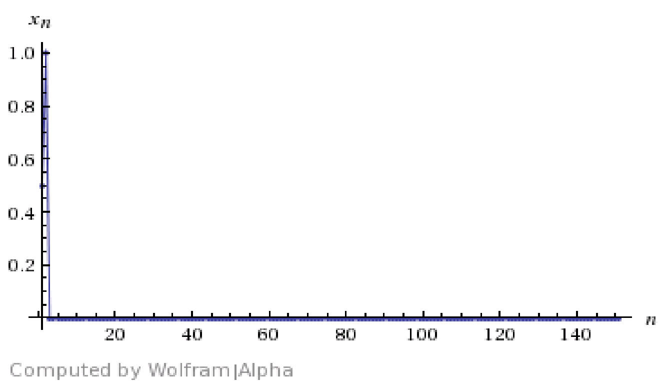

Fig. 8. Graph of $(X) = 4(x_n - x^2{}_n)$, at$x_0 = 0.50000$,

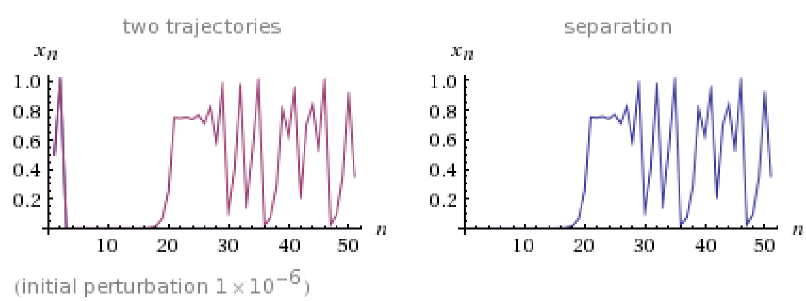

Fig. 9. Unstableness (chaos) of the logistic function at $\lambda = 4$ with $x_0 = 0.50000$

Table 15. The unstable nature of logistic function when $\lambda = 4$ with $x_0 = 0.50000$

Period	Iterates	Linear stability
1	0.00	Unstable
1	0.75	Unstable
2	0.345492, 0.904508	Unstable
3	0.116978, 0.413176, 0.969846	Unstable
3	0.188255, 0.61126, 0.950484	Unstable
4	0.0337639, 0.130496, 0.452866, 0.991487	Unstable
4	0.0432273, 0.165435, 0.552264, 0.989074	Unstable
4	0.277131, 0.801317, 0.636831, 0.925109	Unstable

3 Conclusion

The study has shown that as a system travel through time in dynamical system, the essence of them starting at a particular state and returning back to that state forms what we call recurrent, period-1 recurrent do occur/exist at the attracting fixed point of the logistic function within the ranges $-1 < \lambda < 1$ and $\lambda < 3$ but at the repelling fixed point of the logistic function, period-1 recurrent tends to occur depending on both the control parameter $\lambda\epsilon(0,1) or\ (3,4)$ and the initial point hence when a fixed point is repelling, period-1 recurrent formation is not totally assured. And as the parameter λ is been increases beyond 3 different periodic orbits are formed.

It was also clear that when $\lambda > 3.45$, new periods which show double behaviours are seen and as the control parameter λ is altered beyond 3 and 3.45 periodic doubling which are recurrent are form. We also saw that period-n recurrent might exist since period-3 recurrent formations do exist.

Finally, the unstable behaviour of the logistic function when$\lambda = 4$, ends the periodic behaviour, hence the absence of recurrence. And the absence of recurrent and the unstable nature of system bring about chaos.

Competing Interests

Authors have declared that no competing interests exist.

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